Calculus II: For Physical Sciences and Engineering (AS.110.109)

Lecture: 29

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Textbook Sections: Sorta Chapter 9, sorta some other sources

Today's Topic: Ordinary Differential Equations:

Definition: A **ordinary differential equation** is an equation involving a single variable function $u(x)$ and its derivatives $u'(x)$, $u''(x)$, etc.

For example, the following are all "ODEs" for various functions

1) $u'(t) = ku(t)$ ($u(t)$ is the unknown function, $k \in \mathbb{R}$ fixed)
2) $f'(x) + f''(x) = 0$ ($f(x)$ is the unknown function)
3) $G'''(s) = e^{G(s)} + s$ ($G(s)$ "---")
4) $u'(\theta) + \sqrt{u(\theta) - \sin\theta} = \cos\theta$ ($u(\theta)$ "---")

The **highest order of a derivative** appearing in an ODE is called the **order of the ODE**. 1) and 4) are 1st order ODEs, 2) is a 2nd order ODE, and 3) is a 3rd order ODE.

A **solution** of an ODE is a function which satisfies the relationship dictated by the ODE. For example, $u(x) = e^{kx}$ solves 1), as do $u(x) = 5e^{kx}$ and $u(x) \equiv 0$. So, a given ODE can have many different solutions.

What's the point? The mathematical models of many processes/phenomena in nature are cast in the form of differential equations.

For example:

1) Radioactive decay and population growth can be modelled by 1st order ODEs. We'll see examples in this and the next lecture.
2) Epidemiologists use ODEs to model the process of disease transmission in populations
3) Biologists use ODEs to model the predator-prey relationships between different species.
4) Economists use ODEs to model competition in various industries.

and of course there are **many** more examples.

In higher dimensions, and in the context of functions of more than just one variable (eg. functions of both space and time), models often take the form of **partial differential equations**, which involve derivatives of an unknown function (like $u(x,y,z,t)$) with respect to each of these variables. In this context there is an explosion of applications. As but only three examples:

5) Diffusion processes, including heat flow, are modeled by equations like $\frac{d}{dt}u(x,t) = \frac{d^2}{dx^2}u(x,t)$ (Heat Eqtn) (u = temp at position x at time t)
6) Wave motion is modelled by equations like $\frac{d^2}{dt^2}u(x,t) = \frac{d^2}{dx^2}u(x,t)$ (Wave Eqtn) (u = wave height at position x at time t)
7) The deformation of a load-bearing structural beam is modelled by $\frac{d^2}{dt^2}u(x,t) = -\frac{d^2}{dx^4}u(t,x) + Q(x)$ (Beam Eqtn) (u = vertical displacement of beam from equillibrium at time t under a load $Q(x)$ at position x)

What's the Goal: Say you are a working engineer/researcher/scientist in a field studying some problem/phenomenon.

Step 1: Use principles of the area of interest to cook up a model for the phenomenon – say some differential eqtn (DE) modelling how a quantity of interest changes with respect to eg. space and time.

Step 2: Use calculus/math to analyze the differential eqtn:

A) **Existence Question:** Does the DE actually have solutions? Not every DE has solutions!

B) **Uniqueness Question:** How many solutions are there? We might need to specify "initial" or "boundary" conditions to help isolate a unique solution to the given model.

C) **Stability Question:** If you change the input parameters (eg initial/boundary data), do the solutions change predictably? In the "real world," we must know how to handle measurement errors when apply a mathematical model to make predictions

D) **Numerics/Modelling?** When it comes to interpetting our models and simulating their predictions, we must rely on computers to produce numerical data. Moreover, since many DE models do not produce explicit solutions in terms of elementary functions, it's important to have ways to simulate the solutions in an approximate way.

Step 3: Compare the predictions of the model to the data, and revise/repeat the analysis if necessary.

We will start out in the simplest (but still extremely important) case of **1st order ODEs**:

By a 1st order ODE, we mean an ODE which we can write in the form

$$u'(x) = F(x, u(x))$$

where:
- $F(-,-): \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ is just some function
- $u(x)$ is the unknown function to be solved for

Typically, x is assumed to lie in an appropriate domain, and $u(x)$ might have specified boundary, or initial, conditions.

Examples:

1) $F(x,y) = x+y \longmapsto$ ODE: $u'(x) = F(x,u(x)) = x+u(x)$
2) $F(x,y) = \frac{1}{1+y^2} \longmapsto$ ODE: $u'(x) = \frac{1}{1+u(x)^2}$
3) $F(x,y) = \sin(xy) \longmapsto$ ODE: $u'(x) = \sin(x\cdot u(x))$

In our short foray into ODEs, we will focus on an important, friendlier subclass of ODEs called **linear ODEs**:

Def: An ODE is **linear** if its an equation which is linear in the unknown function and its derivatives. (ie. in 1st order case, $F(x,y)$ is linear in y)

Examples:

1) $u'(x) = \sin(x)\cdot u(x)$
2) $u'(x) = 3u(x) - x$
3) $u'(x) = u(x)^2$ is NOT linear.

Exercises: Before class on friday, classify the order of these ODEs, as well as their linearity/nonlinearity:

1) $y'(x) + y(x) = 0$
2) $g''(x) + g'(x) = 0$
3) $h''(x)\cdot h(x) = 0$
4) $e^x f'(x) + e^{-x} f(x) = 0$
5) $f''(x) + f(x)^2 = 1$
6) $f''''(x) + f(x) = e^x$
7) $u'(x) = \sqrt{1-x^2}$
8) $u''(x) = \sqrt{1-u(x)^2}$
9) $\theta'(x) = \sqrt{1-x^2}\,\theta(x)$.